RADIAL GROWTH OF FUNCTIONS FROM THE KORENBLUM SPACE

A. BORICHEV, YU. LYUBARSKII, E. MALINNIKOVA, AND P. THOMAS

Dedicated to Victor Petrovich Havin on the occasion of his 75th birthday

Abstract. We study radial behavior of analytic and harmonic functions, which admit a certain majorant in the unit disk. We prove that extremal growth or decay may occur only along small sets of radii and give precise estimates of these exceptional sets.

1. Introduction

We study radial behavior of analytic and harmonic functions in the unit disc. In order to describe the problem let us start with the classical results of Lusin and Privalov, see e.g. [12] Ch. IV.

Theorem A. (Lusin, Privalov) *Let $f(z)$ be a function analytic in the unit disc $\mathbf{D}$ and E be a subset of the unit circle $\mathbf{T}$ of positive linear measure. If f tends to zero non-tangentially at each point of E, then $f = 0$.*

The situation changes if one considers radial limits.

Theorem B. (Lusin, Privalov) *There exists an analytic function f in $\mathbf{D}$ such that $\lim_{r\to 1} f(re^{i\phi}) = 0$ for almost every $\phi \in [0, 2\pi)$.*

These results can be reformulated for harmonic functions. The first theorem says that there are no $u \in \mathrm{Harm}(\mathbf{D})$ that tends to $+\infty$ non-tangentially on a set of positive measure in $\mathbf{T}$, while the second gives a function in $\mathrm{Harm}(\mathbf{D})$ that tends radially to $+\infty$ almost everywhere on $\mathbf{T}$ (we remark that the function f in Theorem B can be chosen zero-free). We refer also to [1, 9] for other relevant examples. The growth of harmonic functions tending radially to $+\infty$ almost everywhere can be arbitrarily slow: the statement below is a special case of a theorem in [9].

Theorem C.(Kahane, Katznelson) *Let $v(r)$ be a positive increasing function on $[0, 1)$ and $\lim_{r\to 1-} v(r) = \infty$. Then there exists $u \in \mathrm{Harm}(\mathbf{D})$ such that*

$$u(z) \le v(|z|) \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 31A20 ; Secondary 30H05 .

Key words and phrases. spaces of analytic functions in the disk, harmonic functions, boundary values, Korenblum spaces.

A.B. was partially supported by the ANR project DYNOP, Yu.L. and E.M. were partly supported by the Research Council of Norway, grants 160192/V30 and 177355/V30.

and $\lim_{r\to 1-} u(re^{i\phi}) = \infty$ *for a.e.* $\phi \in [0, 2\pi)$.

In this article we address the following questions.

Let the function v *be as above and* $u \in \mathrm{Harm}(\mathbf{D})$ *satisfy* (1.1).

• *How fast (with respect to* v*) can* u *grow to* $+\infty$ *along massive sets of radii?*

• *How fast (with respect to* v*) can* u *decay to* $-\infty$ *along massive sets of radii?*

We restrict ourselves to a particular majorant function

$$v(r) = \log \frac{1}{1-r}.$$

This choice is motivated by its relevance to the classical Korenblum space $A^{-\infty}$ (see [10]). It also serves as a model case for more general majorants.

The typical answer to the first question is that at almost all radii the function u grows (if it grows at all) slower that v, the exceptional set has zero Lebesgue measure. We give precise estimates on the size of exceptional sets in terms of the Hausdorff measures with respect to the scale of functions $h_\alpha(t) = t|\log t|^\alpha$, $\alpha > 0$.

Regarding the second question we first remark that the function u that satisfies (1.1) may decay to $-\infty$ along radii much faster than $-v(r)$, so the harmonic function $-u$ may fail to satisfy (1.1). However, given $M(s)$, $s > 0$ such that $M(s)/s \to +\infty$, as $s \to +\infty$, the set $\{z \in \mathbf{D} : u(z) < -M(v(|z|)\}$ is small (sharp estimates for typical M are obtained in [3]). We show that along most radii $-u$ grows slower than v, the estimates of the exceptional set being the same as in the answer to the first question. For the maximal possible decay of harmonic functions throughout the whole disc see [11], [2] and references therein.

Our statements can be reformulated for zero-free functions from the Korenblum class. Now the second statement describes how fast an analytic function can approach zero along some radii. Actually this statement holds true for any (non necessarily zero-free) function from $A^{-\infty}$. At the same time adding zeros may result in extremal radial growth along almost all radii.

The paper is organized as follows. The next section includes definitions and formulations of the main results in terms of analytic functions. In Section 3 we deal with harmonic (subharmonic) functions. Using standard estimates of the Poisson integral we show that fast radial growth (decay) implies non-tangential growth (decay) and thus may occur only on a set of zero measure. The main results are proved in Section 4: departing from non-tangential growth (decay) and harmonic measure inequalities we obtain more precise estimates of the size of the exceptional sets. These estimates are sharp, as shown by examples collected in Section 5. We also give an example which shows that the situation becomes very different if one considers growth just along sequences of points: there exists a function such that *every* radius contains a sequence of points of extremal growth. In Section 6 we consider

positive harmonic functions satisfying (1.1) and show that for such functions the answer to the first question is different. Finally, Section 7 contains a theorem about Hausdorff measure of Cantor-type sets.

2. Formulation of the main results

An analytic function f in $\mathbf{D}$ is said to be of class $A^{-\infty}$ if there exist constants C and k such that

$$|f(z)| \le \frac{C}{(1-|z|)^k}.$$

For a function $f \in A^{-\infty}$ we define

$$D_+(f) = \left\{ \theta \in [0, 2\pi) : \liminf_{r\to 1} \frac{\log|f(re^{i\theta})|}{|\log(1-r)|} > 0 \right\}, \tag{2.1}$$

$$D_-(f) = \left\{ \theta \in [0, 2\pi) : \limsup_{r\to 1} \frac{\log|f(re^{i\theta})|}{|\log(1-r)|} < 0 \right\}. \tag{2.2}$$

We recall the definition of the Hausdorff measure. Given an increasing function λ, $\lambda : [0,1) \to [0,+\infty)$, $\lambda(0) = 0$, we denote by $H_\lambda(C)$ the corresponding Hausdorff measure of a set $C \subset \mathbf{T}$ (or $C \subset \mathbf{R}$), which is defined as

$$H_\lambda(C) = \lim_{\epsilon\to 0} \inf \left\{ \sum_s \lambda(|J_s|) : C \subset \cup_s J_s,\ |J_s| < \epsilon \right\},$$

here J_s are arcs of $\mathbf{T}$ (respectively intervals of $\mathbf{R}$).

The main results of the paper give estimates on the size of the sets $D_\pm(f)$.

Theorem 1. *Let $\lambda(t) = o(t|\log t|^\omega)$, $t \to 0$, for every $\omega > 0$. Then*
(i) $H_\lambda(D_+(f)) = 0$ for each zero-free $f \in A^{-\infty}$;
(ii) $H_\lambda(D_-(f)) = 0$ for each $f \in A^{-\infty}$.

These results are sharp as follows from the next statement.

Proposition 1. *For any $\alpha > 0$ there exists a zero-free function $f \in A^{-\infty}$ such that $f^{-1} \in A^{-\infty}$ and $H_\lambda(D_+(f)) = \infty$ for $\lambda(t) = t|\log t|^\alpha$.*

Note that for zero-free functions we have $D_+(f) = D_-(f^{-1})$; thus, Proposition 1 shows that our condition on λ is precise in both assertions of Theorem 1.

There are no analogues of the first estimate in Theorem 1 for general functions from $A^{-\infty}$. This can be seen by analyzing functions having "regular" growth in $\mathbf{D}$ like those given in Theorem 2 in [13]. The argument in [13] relies on the atomization techniques, and in this paper we use a simple explicit function constructed by Horowitz.

Given a number $\mu > 1$ and an integer $\beta > 1$, consider the function

$$f_{\mu,\beta}(z) = \prod_{k=1}^{\infty} \left(1 + \mu z^{\beta^k}\right). \tag{2.3}$$

It follows from [8] (see also [10]) that $f_{\mu,\beta} \in A^{-\infty}$.

Proposition 2. *Let the numbers μ and β satisfy the conditions*

$$1 - \frac{1}{\mu^{1-1/\sqrt{\beta}}} \geq \frac{1}{e}, \quad \mu > e. \tag{2.4}$$

Then $D_+(f_{\mu,\beta})$ has full measure in $\mathbf{T}$.

We also consider the extremal growth on subsets of radii. Given a function $f \in A^{-\infty}$, denote

$$G_+(f) = \left\{ \theta \in [0, 2\pi) : \limsup_{r \to 1} \frac{\log |f(re^{i\theta})|}{|\log(1-r)|} > 0 \right\}$$

Clearly $G_+(f) \supset D_+(f)$.

Proposition 3. *There exists a zero-free $f \in A^{-\infty}$, such that $G_+(f) = \mathbf{T}$.*

The estimate in the first statement in Theorem 1 can be improved if we assume that $|f|$ is bounded from below by a positive constant. This improvement corresponds to the difference between estimates of the Poisson integral with respect to a premeasure (as in Theorem 1) and a Borel measure as in Theorem 2 below; we refer the reader to [10] for the definition and basic properties of premeasures.

Theorem 2. *Let $\lambda(t) = t|\log t|$. Suppose that $f \in A^{-\infty}$ and $|f|$ is bounded from below by a positive constant. Then the set $G_+(f)$ is a countable union of sets with finite H_λ measure.*

There exists $f \in A^{-\infty}$, such that $|f|$ is bounded form below by a positive constant and $H_\lambda(D_+(f)) = \infty$.

In order to construct examples in Proposition 1 and Theorem 2 we use Cantor-type sets having the following structure:

$$C = \cap_s C_s, \ C_s \supset C_{s+1}, \ C_0 = [0, 1],$$

each set C_s is a union of N_s segments $\{I_j^{(s)}\}_j$ of the same length l_s. For each such segment the intersection $C_{s+1} \cap I_j^{(s)}$ is a union of k_s disjoint segments of length l_{s+1}. We assume, of course, that

$$l_s \searrow 0, \ s \to \infty; \ k_s l_{s+1} < l_s, \text{ and } N_s = k_0 k_1 \dots k_{s-1}.$$

Theorem 3. *Let $\lambda : [0, 1) \to [0, +\infty)$ be a continuous increasing function with $\lambda(0) = 0$, such that for some $a > 0$ and $s > s_0$*

$$\frac{\lambda(l)}{l} \geq a \frac{\lambda(l_{s+1})}{l_{s+1}} \quad \text{for any } l \in [l_{s+1}, l_s). \tag{2.5}$$

Then

$$\liminf_{s \to \infty} N_s \lambda(l_s) \geq H_\lambda(C) \geq \frac{a}{2} \liminf_{s \to \infty} N_s \lambda(l_s). \tag{2.6}$$

Other results of such type are given in [5, 4]; unfortunately, we are not able to apply those results in our situation. We prove Theorem 3 in the last section and believe that it may be of its own interest.

3. From radial to non-tangential growth

To deal with zero-free functions from $A^{-\infty}$ we consider the corresponding class of harmonic functions. A function $u \in \mathrm{Harm}(\mathbf{D})$ is said to be of class $\mathcal{K}$ if there exists a constant C such that

$$u(z) \le C \log \frac{e}{1-|z|}, \qquad z \in \mathbf{D}.$$

If $f \in A^{-\infty}$ is a zero-free function, then $u(z) = \log|f(z)|$ belongs to $\mathcal{K}$.

Given a function $u \in \mathcal{K}$ we denote

$$E_+(u) = \left\{ \theta \in [0, 2\pi) : \liminf_{r \to 1} \frac{u(re^{i\theta})}{|\log(1-r)|} > 0 \right\},$$

$$E_-(u) = \left\{ \theta \in [0, 2\pi) : \limsup_{r \to 1} \frac{u(re^{i\theta})}{|\log(1-r)|} < 0 \right\}.$$

In this section we first show that fast radial growth along radii implies fast non-tangential growth. We use the standard notation

$$P(re^{i\theta}) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

for the Poisson kernel.

Let $r \in (0,1)$, $\tau \in (0,1)$, and $0 < \delta < \tau(1-r)$. Then

$$P(re^{i\theta}) > (1-\tau) P(re^{i(\theta+\delta)}) \tag{3.1}$$

This inequality can be proved by elementary calculations, it can be also viewed as a special case of the Harnack inequality.

Lemma 1. *Let $u \in \mathcal{K}$, $C = \sup\{u(z)\,(\log e/(1-|z|))^{-1}\,; z \in \mathbf{D}\}$. Suppose that for some $\sigma > 0$, $\theta \in [0, 2\pi]$, and $r \in (0,1)$*

$$u(re^{i\theta}) > \sigma \log \frac{e}{1-r}.$$

Then

$$u(re^{i(\theta+\delta)}) > \frac{\sigma}{2} \log \frac{e}{1-r} \tag{3.2}$$

for $|\delta| < \tau_1(1-r)$, where $\tau_1 = \tau_1(C, \sigma) > 0$.

Proof. Let $R=(1+r)/2$. We apply (3.1), replacing $P(r,\cdot)$ by $P(r/R,\cdot)$ and assuming that $|\delta|<\tau\left(1-\frac{r}{R}\right)$ with $\tau<1$. We obtain

$$u(re^{i\theta})=\int_0^{2\pi}u(Re^{i\phi})P\left(\frac{r}{R}e^{i(\theta-\phi)}\right)d\phi=(1-\tau)u(re^{i(\theta+\delta)})+$$
$$\int_0^{2\pi}u(Re^{i\phi})\left(P\left(\frac{r}{R}e^{i(\theta-\phi)}\right)-(1-\tau)P\left(\frac{r}{R}e^{i(\theta+\delta-\phi)}\right)\right)d\phi<$$
$$(1-\tau)u(re^{i(\theta+\delta)})+C\tau\log\frac{e}{1-R}.$$

Hence

$$\sigma\log\frac{e}{1-r}<(1-\tau)u(re^{i(\theta+\delta)})+C\tau\log\frac{e}{1-r}+C\tau\log 2.$$

Taking τ small enough we now obtain relation (3.2) with $\tau_1=\tau/2$. □

The proof of Lemma 1 works also for the radial decay; however, for this case we need a more general setting involving subharmonic functions.

Lemma 2. *Let v be a subharmonic function on $\mathbf{D}$ and*

$$v(z)\le C\log\frac{e}{1-|z|},\ z\in\mathbf{D}.$$

Suppose that for some $\sigma>0$, $\theta\in[0,2\pi]$, and $r_0\in(0,1)$

$$v(re^{i\theta})<-\sigma\log\frac{e}{1-r},\ r>r_0$$

Then

$$v(re^{i(\theta+\delta)})<-\frac{\sigma}{2}\log\frac{e}{1-r},\ r>r_1$$

for $|\delta|<\tau_2(1-r)$, where $\tau_2=\tau_2(C,\sigma)>0$, $r_1=r_1(r_0)<1$.

Proof. Without loss of generality assume that $\theta=0$. Consider the function

$$w(\rho e^{i\varphi})=v(1-\rho e^{i\varphi})$$

which is subharmonic in the domain

$$G=\{\zeta=\rho e^{i\varphi}:0<\rho<1,0<\varphi<\pi/4\}.$$

We have

$$w(s)<-\sigma\log\frac{e}{s},\quad s<\rho_0,\ \text{and}\ w(se^{i\varphi})<C\log\frac{e}{s},\ se^{i\varphi}\in G, \tag{3.3}$$

and we need to prove that

$$w(\rho e^{i\delta})<-\frac{\sigma}{2}\log\frac{e}{\rho},\ \rho<\rho_1\ \text{for all}\ \delta\in(0,\delta_1),\ \delta_1=\delta_1(\sigma,C). \tag{3.4}$$

Consider an auxiliary function $u(\zeta)$ which is harmonic in the domain $R=\{\zeta=\rho e^{i\varphi}:1/4<|\rho|<1,\ 0<\varphi<\pi/4\}$ and has the boundary values

$$u(\rho)=0,\ \rho\in(1/4,1),\ u(\zeta)|_{\partial R\setminus(1/4,1)}=1.$$

Fix now $\rho < \rho_0/2$. It follows from (3.3) that

$$w(\zeta) + \sigma \log \frac{e}{\rho} \le (C+\sigma) u\left(\frac{\zeta}{2\rho}\right) \log \frac{e}{\rho} + \sigma \log 2, \quad \zeta \in \partial(2\rho R).$$

Therefore

$$w(\zeta) < -\sigma \log \frac{e}{\rho} + (C+\sigma) u\left(\frac{\zeta}{2\rho}\right) \log \frac{e}{\rho} + \sigma \log 2, \quad \zeta \in 2\rho R.$$

In order to obtain (3.4) it remains to take $\zeta = \rho e^{i\varphi}$ and note that $u(e^{i\varphi}/2) \to 0$ as $\varphi \to 0$. $\square$

It follows from Lemmas 1, 2, and the Lusin-Privalov theorem, that $|E_+(u)| = 0$ for any $u \in \mathcal{K}$ and also $|D_-(f)| = 0$ for any $f \in A^{-\infty}$.

4. Proof of Theorem 1

In this section we prove Theorem 1. The first statement of the theorem is equivalent to the following

Theorem 1 *i*. *Let $\lambda(t) = o(t|\log t|^\omega)$, $t \to 0$, for every $\omega > 0$. Then $H_\lambda(E^+(u)) = 0$ for every $u \in \mathcal{K}$.*

Proof. Fix $u \in \mathcal{K}$. Let

$$E_n = \left\{ e^{i\theta} : u(re^{i\theta}) \ge \frac{1}{n} \log \frac{e}{1-r},\ r \ge 1 - \frac{1}{n} \right\}.$$

Since $E_+(u) = \cup_n E_n$, it suffices to prove that $H_\lambda(E_n) = 0$ for each n. We use the standard cone construction. For $e^{i\theta} \in \mathbf{T}$ and $a < 1$ consider the Stolz angle $\Gamma^a_\theta = \mathrm{conv}(e^{i\theta}, a\mathbf{D})$, i.e., the convex hull of $e^{i\theta}$ and the disc of radius a. By Lemma 1, one can find $a > 0$ such that $u(z) \ge \frac{1}{2n} \log \frac{e}{1-|z|}$ for all $e^{i\theta} \in E_n$ and $z \in \Gamma^\theta_a$, $|z| > 1 - \frac{1}{n}$.

Let

$$\Omega = \cup_{\theta \in E_n} \Gamma^\theta_a.$$

The function u is bounded from below on Ω; let, say, $u \ge c_0$. Let $t < 1$ be sufficiently close to 1 and

$$\Omega_t = \Omega \cap t\mathbf{D}.$$

For an appropriate $b = b(a)$ we have

$$\partial\Omega_t \cap t\mathbf{T} = tE_n^{b(1-t)} = \{te^{i\theta} : |\theta - \theta_0| < b(1-t) \text{ for some } e^{i\theta_0} \in E_n\}.$$

Estimating the subharmonic function u in the domain Ω_t, $t > 1 - \frac{1}{n}$, in terms of harmonic measure, we obtain

$$u(0) \ge c_1 + \omega(0, tE_n^{b(1-t)}, \Omega_t) \frac{1}{2n} \log \frac{e}{1-t}. \tag{4.1}$$

Domains Ω_t have Lipschitz boundaries with Lipschitz constants bounded uniformly in t. By the Lavrentiev theorem, (see e.g. [6], Chapter VII,

Theorem 4.3), there exist c and γ such that, for each subarc $I \subset \partial\Omega_t$ and $A \subset I$, we have

$$\frac{\omega(0, A, \Omega_t)}{\omega(0, I, \Omega_t)} \geq c \left(\frac{l(A)}{l(I)}\right)^{\gamma},$$

here l is the arc-length on $\partial\Omega_t$. In particular, by (4.1),

$$l(tE_n^{b(1-t)})^{\gamma} \leq c^{-1} l(\partial\Omega_t)^{\gamma} \omega(0, tE_n^{b(1-t)}, \Omega_t) \leq C \left(\log \frac{e}{1-t}\right)^{-1},$$

where $C = C(n)$ does not depend on t. Hence for all $\epsilon > 0$ small enough we have

$$l(E_n^{\epsilon}) \leq C \left(\log \frac{be}{\epsilon}\right)^{-1/\gamma}.$$

Therefore one can cover E_n by N_ϵ intervals of length ϵ, with

$$N_\epsilon \leq 2\epsilon^{-1} C \left(\log \frac{be}{\epsilon}\right)^{-1/\gamma}.$$

Then

$$H_\lambda(E_n) \leq \liminf_{\epsilon \to 0} N_\epsilon \lambda(\epsilon) \leq \liminf_{\epsilon \to 0} 2\epsilon^{-1} C \left(\log \frac{be}{\epsilon}\right)^{-1/\gamma} \lambda(\epsilon).$$

The condition on λ implies $H_\lambda(E_n) = 0$ and we are done. □

To prove the second part of Theorem 1 we repeat the argument for the subharmonic function $v(z) = \log |f(z)|$, using Lemma 2 instead of Lemma 1, and replace the inequality (4.1) by the following estimate, valid for subharmonic functions:

$$v(0) \leq \int_{\partial\Omega_t} v(z) d\omega(0, z, \Omega_t) \leq c_1 - \omega(0, tE_n^{b(1-t)}, \Omega_t) \frac{1}{2n} \log \frac{e}{1-t}.$$

5. Sharpness of results

First we construct functions from $\mathcal{K}$ with "large" sets of growth.

Lemma 3. *For each integer $A \geq 2$ the series*

$$u(z) = \Re \sum_{k=0}^{\infty} A^k z^{2^{A^k}}$$

converges in $\mathbf{D}$ *and* $|u(z)| \leq C \log \frac{e}{1-|z|}$.

Proof. Fix $z \in \mathbf{D}$ sufficiently close to the boundary, and choose N such that

$$2^{-A^N} \geq 1 - |z| > 2^{-A^{N+1}}. \tag{5.1}$$

Then

$$-\log |z| = -\log(1 - (1 - |z|)) \geq 1 - |z| > 2^{-A^{N+1}},$$

and for $n \geq N + 1$ we have

$$\frac{A^{n+1}|z|^{2^{A^{n+1}}}}{A^n|z|^{2^{A^n}}} = A|z|^{2^{A^{n+1}}-2^{A^n}} \le Ae^{-2^{-A^{N+1}}\left(2^{A^{n+1}}-2^{A^n}\right)} < \delta(A) < 1,$$

with

$$\lim_{A\to\infty} \delta(A) = 0. \tag{5.2}$$

Therefore

$$|u(z)| \le \sum_{n=0}^{N} A^n|z|^{2^{A^n}} + \sum_{n=N+1}^{\infty} A^n|z|^{2^{A^n}} \le$$

$$A^{N+1} + A^{N+1}\frac{1}{1-\delta(A)} \le \frac{2}{1-\delta(A)}\frac{A}{\log 2}\log\frac{1}{1-|z|}.$$

□

Proof of Proposition 1. Let A be large enough and

$$f(z) = \exp\left(\sum_{k=1}^{\infty} A^k z^{2^{A^k}}\right).$$

Then $u = \log|f|$ is the function from the previous Lemma, hence both f and f^{-1} are from $A^{-\infty}$. If, for some ϕ, we have $\cos(2^{A^k}\phi) \ge 1/\sqrt{2}$ for each k, then $\Re\Big(\big(re^{i\phi}\big)^{2^{A^k}}\Big) \ge r^{2^{A^k}}/\sqrt{2} > 0$. Taking N as in (5.1), with $z = re^{i\phi}$, we get

$$u(re^{i\phi}) \ge \frac{1}{2}A^N r^{2^{A^N}} \ge \frac{1}{8}A^N \ge \frac{1}{8A\log 2}\log\frac{1}{1-r},$$

thus $\phi \in E_+(u)$. Denote

$$C_j = \cap_{k=0}^{j}\{\phi : \cos(2^{A^k}\phi) \ge 1/\sqrt{2}\},\ C = \cap C_j.$$

Then C_j is the union of N_j intervals of length $l_j = c2^{-A^j}$, where c is an absolute constant. We call them intervals from j-th generation. Each of them contains $k_{j+1} = c2^{A^{j+1}-A^j}$ intervals from the next generation. So it is easy to see that $N_j = c^j 2^{A^j}$. Theorem 3 with $\lambda(t) = t|\log t|^\alpha$ now yields

$$H_\lambda(E_+(u)) \ge H_\lambda(C) \ge \frac{c_1}{2A^\alpha}\liminf_j c^j A^{\alpha j}.$$

We chose A such that $A^\alpha > c^{-1}$ and obtain $H_\lambda(E_+(u)) = +\infty$. □

Next we construct an auxiliary harmonic function.

Lemma 4. *There exist a function $h \in \mathrm{Harm}(\mathbf{D})$ and a positive B such that $|h(z)| \le B|z|$, $z \in \mathbf{D}$, and*

$$\max_{1/6<r<1/3} h(re^{i\theta}) \ge 1, \qquad \theta \in [0, 2\pi).$$

Proof. Set $K=\{\frac{t+1}{6}e^{3\pi ti}, 0\le t\le 1\}$. Let f be a function equal to 0 in a small neighborhood of 0 and to 2 in a small neighborhood of K. By the Runge theorem, we can approximate f by a polynomial g in such a way that $|g-f|<\frac{1}{3}$ on $K\cup\{0\}$. Then we can just set $h=\Re(g-g(0))$. $\square$

Proposition 3 follows immediately from the following

Lemma 5. *If an integer A is large enough, then the series*

$$u(z)=\sum_{k=0}^{\infty}A^k h\big(z^{2^{A^k}}\big)$$

converges in $\mathbf{D}$ to a function from $\mathcal{K}$, and for some $d>0$,

$$\limsup_{r\to 1}\frac{u(re^{i\theta})}{|\log(1-r)|}\ge d,\qquad \theta\in[0,2\pi).$$

Proof. Fix $z\in\mathbf{D}$ sufficiently close to the boundary, and choose N such that

$$2^{-A^N}\ge 1-|z|>2^{-A^{N+1}}.$$

By (5.2),

$$|u(z)|\le B\sum_{n=0}^{N}A^n|z|^{2^{A^n}}+B\sum_{n=N+1}^{\infty}A^n|z|^{2^{A^n}}\le$$
$$A^{N+1}B+A^{N+1}B\frac{1}{1-\delta(A)}\le\frac{2}{1-\delta(A)}\frac{AB}{\log 2}\log\frac{1}{1-|z|}.$$

The same estimate gives

$$|u(z)-A^{N+1}h(z^{2^{A^{N+1}}})|\le\frac{A^{N+1}B}{A-1}+A^{N+1}B\sum_{k\ge 1}\delta(A)^k\le\frac{A^{N+1}}{2}$$

for large A.

Finally, given $\theta\in[0,2\pi)$, we construct a sequence of points $\{w_N=|w_N|e^{i\theta}\}$ at which u has extremal growth.

Let $r_N(\theta)\in(\frac{1}{6},\frac{1}{3})$ be such that

$$h\big(r_N(\theta)e^{i\theta\cdot 2^{A^{N+1}}}\big)\ge 1,$$

and let $w_N=r_N(\theta)^{2^{-A^{N+1}}}e^{i\theta}$. Then

$$2^{-A^N}\ge 1-|w_N|>2^{-A^{N+1}},$$

and

$$u(w_N)\ge A^{N+1}h(w_N^{2^{A^{N+1}}})-\frac{A^{N+1}}{2}=A^{N+1}h(r_N(\theta)e^{i\theta\cdot 2^{A^{N+1}}})-\frac{A^{N+1}}{2}$$
$$\ge\frac{A^{N+1}}{2}\ge c\log\frac{1}{1-|w_N|}.$$

$\square$

Proof of Proposition 2. Let now $\mu > 1$ and an integer $\beta > 1$ satisfy (2.4) and $f_{\mu,\beta}$ be the corresponding Horowitz function, see (2.3). The zeros of $f_{\mu,\beta}$ are of the form

$$z_{k,j} = \rho_k e^{i\pi\beta^{-k}} e^{2i\pi j\beta^{-k}}, \quad k = 1, 2, \dots, \quad j = 0, 1, \dots, \beta^k - 1, \tag{5.3}$$

where $\rho_k = \mu^{-\beta^{-k}}$.

We will construct a sequence $\{\epsilon_k\}$ such that $\epsilon_k \searrow 0$,

$$\sum_{k=1}^{\infty} \beta^k \epsilon_k < \infty, \tag{5.4}$$

and, for some $a > 0$,

$$|f_{\mu,\beta}(z)| \ge \frac{\text{Const}}{(1-|z|)^a}, \text{ for } z \notin \cup_{k,j} D_{k,j}, \tag{5.5}$$

here $D_{k,j} = \{z \in \mathbf{D}; |z - z_{k,j}| < \epsilon_k\}$. Since $\rho_k = \mu^{-\beta^{-k}}$, the discs $D_{k,j}$ are contained in the open unit disc; by (5.4), the sum of the lengths of the projections of $\cup_j D_{k,j}$ on $\mathbf{T}$ is finite. Now Proposition 2 follows readily.

Consider the circle $|z| = r < 1$ and choose an integer m such that

$$r^{\beta^m} \ge \mu^{-\sqrt{\beta}} > r^{\beta^{m+1}}, \tag{5.6}$$

and hence,

$$\mu r^{\beta^{m-1}} \ge \mu^{1-1/\sqrt{\beta}} > 1 \quad \text{and} \quad \mu r^{\beta^{m+1}} < \mu^{1-\sqrt{\beta}} < 1.$$

Relation (5.6) also yields

$$\left| m - \frac{1}{\log\beta} \log\frac{1}{1-r} \right| < C, \tag{5.7}$$

where the constant C does not depend on r. One may assume $m > 1$. We have

$$f(z) = \underbrace{\prod_{k=1}^{m-1} \left(1 + \mu z^{\beta^k}\right)}_{P_m(z)} \left(1 + \mu z^{\beta^m}\right) \underbrace{\prod_{k=m+1}^{\infty} \left(1 + \mu z^{\beta^k}\right)}_{R_m(z)}. \tag{5.8}$$

The first factor is uniformly large on the circle $|z| = r$:

$$\log|P_m(z)| \ge \sum_{k=1}^{m-1} \log\left(\mu r^{\beta^k} - 1\right) =$$
$$\sum_{k=1}^{m-1} \left(\log\mu + \beta^k \log r\right) + \sum_{k=1}^{m-1} \log\left(1 - \frac{1}{\mu r^{\beta^k}}\right).$$

We get

$$\sum_{k=1}^{m-1}\Big(\log\mu+\beta^k\log r\Big)\ge(m-1)\log\mu+\beta^m\log r$$
$$\ge(m-1)\log\mu-\sqrt{\beta}\log\mu$$

and

$$\sum_{k=1}^{m-1}\log\left(1-\frac{1}{\mu r^{\beta^k}}\right)\ge\sum_{k=1}^{m-1}\log\left(1-\frac{1}{\mu r^{\beta^{m-1}}}\right)$$
$$\ge(m-1)\log\left(1-\frac{1}{\mu^{1-1/\sqrt{\beta}}}\right).$$

Relation (2.4) now yields

$$\log|P_m(z)|\ge(m-1)(\log\mu-1)-\sqrt{\beta}\log\mu,$$

and, by (5.7),

$$|P_m(z)|\ge\frac{\text{Const}}{(1-|z|)^a},\qquad |z|=r,\ a=\frac{\log\mu-1}{\log\beta}.$$

We apply the inequality $\log(1-x)\ge-\alpha x$, $x\le1-\alpha^{-1}$ in order to prove that the third factor in (5.8) is separated from zero when $|z|=r$:

$$\log|R_m(z)|\ge\sum_{k=m+1}^{\infty}\log\Big(1-\mu|z|^{\beta^k}\Big)\ge-\alpha\sum_{k=m+1}^{\infty}\mu|z|^{\beta^k}\ge c(\mu,\beta),$$

if we take $\alpha=\Big(1-\mu^{1-\sqrt{\beta}}\Big)^{-1}$, say.

The second factor in (5.8) vanishes at the points $\{z_{m,j}\}$, $j=0,1,\dots,\beta^m-1$. Fix now $q<1$ and let $\epsilon_m=q^m\beta^{-m}$. Condition (5.4) is then fulfilled. It is also straightforward that

$$|1+\mu z^{\beta^m}|\ge cq^m$$

when $|z-z_{m,j}|=\epsilon_m$ for some j. Then the minimum principle implies the same inequality whenever $\operatorname{dist}(z,\{z_{m,j}\}_j)>\epsilon_m$.

It follows now from (5.7) that, for any $a'<a$, one can chose q sufficiently close to 1 such that

$$|f_{\mu,\beta}(z)|\ge\frac{\text{Const}}{(1-|z|)^{a'}},\ \text{for } z\notin\cup_{k,j}D_{k,j}.$$

This completes the proof of Proposition 2 $\square$

6. Positive harmonic functions

In this section we prove Theorem 2. First we prove that given a positive function $u \in \mathcal{K}$, the set

$$F_+(u) = \left\{\theta \in [0, 2\pi) : \limsup_{r \to 1} \frac{u(re^{i\theta})}{|\log(1-r)|} > 0\right\}$$

is a countable union of sets with finite H_λ measure, with the measuring function $\lambda(t) = t|\log t|$. This implies the first statement of Theorem 2.

Let

$$F_n = \left\{\theta \in [0, 2\pi) : \limsup_{r \to 1} \frac{u(re^{i\theta})}{|\log(1-r)|} \geq \frac{2}{n}\right\}.$$

It suffices to prove that $H_\lambda(F_n) < \infty$ for all n.

The function u is positive and harmonic so it is the Poisson integral of a finite measure μ on $\mathbf{T}$. Since $u \in \mathcal{K}$ we have

$$\mu(I) \leq C|I| \log \frac{e}{|I|} \tag{6.1}$$

for any arc I on the unit circle (see [10]).

In what follows we denote

$$\mu(\alpha, \beta) = \mu(\{e^{i\varphi}; \alpha \leq \varphi < \beta\}).$$

Lemma 6. *For each n and each $\theta \in F_n$ there exists a decreasing sequence $\{\Delta_j\}$, $\Delta_j \to 0$ as $j \to \infty$ which satisfies*

$$\mu(\theta - \Delta_j, \theta + \Delta_j) \geq k\left(10\Delta_j \log \frac{1}{10\Delta_j}\right), \tag{6.2}$$

with some $k > 0$, depending on C and n only.

Suppose this lemma is already proved. For each $\epsilon > 0$ we can cover F_n by intervals I with centers on F_n and of length less than ϵ which satisfy $\mu(I) \geq k|5I|\big|\log|5I|\big|$, where $5I$ is the interval concentric with I of length 5 times that of I. By the Vitali lemma (see, for example, [7, page 2]) we can find a subfamily I_j of disjoint intervals such that $F_n \subset \cup_j 5I_j$. We obtain

$$\sum_j |5I_j||\log|5I_j| \leq \frac{1}{k}\sum_j \mu(I_j) \leq \frac{1}{k}\mu(\mathbf{T}),$$

which yields $H_\lambda(F_n) \leq \frac{1}{k}\mu(\mathbf{T}) < +\infty$.

Proof of Lemma 6. For $\theta \in F_n$ there exists a sequence $\{r_j\}_1^\infty$ such that $r_j \nearrow 1$ and

$$\frac{1}{n}\log\frac{1}{1-r_j} \leq u(r_j e^{i\theta}) = \int_{-\pi}^{\pi} P(r_j e^{i\phi}) d\mu(\theta - \phi). \tag{6.3}$$

Let a, A be two constants, such that $0 < a < A$, their values will be determined below and $\delta_j = a(1-r_j)$, $\Delta_j = A(1-r_j)$. By choosing a sufficiently small and using (6.1) we can achieve

$$\int_{-\delta_j}^{\delta_j} P(r_j e^{i\phi}) d\mu(\theta-\phi) \le \frac{1}{10n}\log\frac{1}{1-r_j},\ j>j_0. \tag{6.4}$$

Furthermore, let

$$Q(re^{i\phi}) = -\partial_\phi P(re^{i\phi}) = \frac{1}{2\pi}\frac{2r(1-r^2)\sin\phi}{(1-2r\cos\phi+r^2)^2}.$$

be the angular derivative of the Poisson kernel. We then have

(6.5)

$$\int_{\delta_j<|\phi|\le\pi} P(r_je^{i\phi})d\mu(\theta-\phi) \le \mu(\mathbf{T}) + \int_{\delta_j}^{\pi}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi =$$

$$\mu(\mathbf{T}) + \int_{\delta_j}^{\Delta_j}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi + \int_{\Delta_j}^{\pi}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi.$$

In addition,

$$\int_{\Delta_j}^{\pi}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi \le C\log\frac{e}{2\Delta_j}\int_{\Delta_j}^{\pi}2\phi Q(r_je^{i\phi})d\phi \le$$

$$2C\log\frac{1}{1-r_j}\left(A(1-r_j)P\left(r_je^{iA(1-r_j)}\right) + \int_{A(1-r_j)}^{\pi}P(r_je^{i\phi})d\phi\right).$$

Taking A sufficiently large we obtain

$$\int_{\Delta_j}^{\pi}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi \le \frac{1}{10n}\log\frac{1}{1-r_j},\ j>j_0. \tag{6.6}$$

It follows now from (6.3), (6.4), (6.5), and (6.6) that

$$\int_{\delta_j}^{\Delta_j}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi > \frac{1}{5n}\log\frac{1}{1-r_j},\ j>j_0. \tag{6.7}$$

Integration by parts gives

$$\int_{\delta_j}^{\Delta_j}\phi Q(r_je^{i\phi})d\phi \le \int_0^{\pi}\phi Q(r_je^{i\phi})d\phi \le \int_0^{\pi}P(r_je^{i\phi})d\phi = \frac{1}{2}.$$

This together with (6.7) implies

$$\int_{\delta_j}^{\Delta_j}\mu(\theta-\phi,\theta+\phi)Q(r_je^{i\phi})d\phi > \frac{1}{5n}\log\frac{1}{1-r_j}\int_{\delta_j}^{\Delta_j}\phi Q(r_je^{i\phi})d\phi,\ j>j_0.$$

Therefore, for each $j>j_0$ there exists $\phi_j\in(\delta_j,\Delta_j)$ such that

$$\mu(\theta-\phi_j,\theta+\phi_j) > \frac{\phi_j}{5n}\log\frac{1}{1-r_j}.$$

The desired estimate (6.2) follows. $\square$

To complete the proof of Theorem 2 we need to construct a positive harmonic function $u \in \mathcal{K}$ with $H_\lambda(E_+(u)) = \infty$, where $\lambda(t) = t|\log t|$. Taking then its harmonic conjugate $\tilde{u}$ we obtain the desired function as $f = \exp(u + i\tilde{u})$. Clearly $D_+(f) = E_+(u)$, $f \in A^{-\infty}$, and $|f| \geq 1$.

First we construct a function $v \in \mathcal{K}$ such that $H_\lambda(E_+(v)) > 0$. We use a Cantor-type construction.

Let C_1 be the union of two opposite quarters of the circle. We construct by induction sets $C_k \subset C_{k-1}$ such that C_k consists of $2^{2^k - k}$ closed arcs of length $2^{1-2^k}\pi$ each. To obtain C_k we divide each of the arcs of C_{k-1} into $2^{2^{k-1}}$ equal subarcs and choose each second of them for C_k. Denote $C = \cap C_k$ and consider the measures $d\mu_k = 2^k \chi(C_k)dt$, where $\chi(C_k)$ is the characteristic function of C_k.

Lemma 7. *The sequence $\{\mu_k\}$ converges weakly to a measure μ_0 and $v = P * \mu_0 \in \mathcal{K}$. In addition $C \subset E_+(v)$ and $H_\lambda(C) > 0$.*

Proof. We note that $\mu_k(\mathbf{T}) = 2\pi$ for each k. Besides, for each arc I, with endpoints of the form $\exp(2\pi m i 2^{-2^s})$, where m is integer, the limit $\mu_k(I)$ as $k \to \infty$ exists, just because all values $\mu_k(I)$ are the same when $k > s$. Now each continuous function on the circle can be uniformly approximated by linear combinations of characteristic functions of such dyadic arcs. Thus for each continuous function f there exists

$$\lim_{k\to\infty} \int_{\mathbf{T}} f d\mu_k$$

and μ_k converge weakly to some positive measure μ_0.

In order to prove that $v = P * \mu_0 \in \mathcal{K}$ it suffices to check that

$$\mu_0(J) \leq \text{const}|J| \log \frac{1}{|J|}$$

for each arc $J \subset \mathbf{T}$, and then to use again the results from [10].

Choose s such that $2^{-2^s} 2\pi < |J| \leq 2^{-2^{s-1}} 2\pi$. Now take an arc $J_0 \supset J$ with endpoints of the form $\exp(2\pi m i 2^{-2^s})$ with integer m and such that $|J_0| < 3|J|$. We obtain

$$\mu_k(J) \leq \mu_k(J_0) = \mu_s(J_0) \leq 2^s |J_0| < 6|J|| \log |J||.$$

which is the desired inequality.

We now check that $C \subset E_+(v)$. We have

$$v(re^{i\alpha}) = \int_{-\pi}^{\pi} P(re^{i\phi}) d\mu_0(\alpha - \phi) \geq \int_0^{\pi} \mu_0(\alpha - \phi, \alpha + \phi) Q(re^{i\phi}) d\phi.$$

Let $\alpha \in \cap C_k = C$ and $2^{k-1} \leq |\log(1-r)| < 2^k$. Then
$\mu_0(\alpha - \phi, \alpha + \phi) \geq c2^k \phi$ for $\phi < 1 - r$ and

$$v(re^{i\alpha}) \geq \int_0^{1-r} \mu_0(\alpha - \phi, \alpha + \phi) Q(re^{i\phi}) d\phi \geq c2^k \int_0^{1-r} \phi Q(re^{i\phi}) d\phi \geq c_1 2^k,$$

when $r > r_0$. Thus $C = \cap C_k \subset E_+(v)$. Remind that C_k is the union of $2^{2^k - k}$ arcs of length $2\pi 2^{-2^k}$ and C is a set of the type described in Theorem 3. For $\lambda(t) = t|\log t|$ the theorem gives $H_\lambda(C) \ge c > 0$. $\square$

Finally we construct a sequence of measures $\mu^{(n)}$ and sets $C^{(n)}$ such that $v^{(n)} = P * \mu^{(n)}$ is in $\mathcal{K}$, $E_+(v^{(n)}) \supset C^{(n)}$, $H_\lambda(C^{(n)}) \to \infty$ as $n \to \infty$.

The construction of $C^{(n)}$ is the following. Let $C_1^{(n)}$ be the union of 2^n arcs of length $2\pi 2^{-n-1}$ (we divide the circle into 2^{n+1} equal arcs and take each second), $\mu_1^{(n)} = 2^{-n+1}\chi(C_1^{(n)})$. Let $C_k^{(n)}$ be the union of $2^{2^{k-1}(n+1)-k}$ arcs of length $2\pi 2^{-2^{k-1}(n+1)}$, then we divide each arc into equal arcs of length $2\pi 2^{-2^k(n+1)}$ and take each second of them to form C_{k+1}. We define also $\mu_k^{(n)} = 2^{-n+k}\chi(C_k^{(n)})$. As earlier the sequence of measures $\mu_k^{(n)}$ converges to a singular measure $\mu^{(n)}$ such that

$$\mu^{(n)}(J) \le \text{const} \cdot 2^{-n} \cdot |J| \log \frac{1}{|J|}$$

for every $n \ge 1$ and for every arc $J \subset \mathbf{T}$. Then $v^{(n)} = P * \mu^{(n)} \in \mathcal{K}$, $u = \sum_{n\ge 1} v^{(n)} \in \mathcal{K}$, and $E_+(v^{(n)}) \supset C^{(n)}$. Theorem 3 shows that $H_\lambda(C^{(n)}) \ge cn$, and, hence, $H_\lambda(E_+(u)) = \infty$.

7. Hausdorff measure of Cantor sets

In this section we prove Theorem 3. The left hand side inequality in (2.6) is straightforward.

We say that C_s is the set of s'th generation, and the intervals $I_i^{(s)}$ of length l_s that constitute C_s are the intervals of s'th generation. Denote the set of all these intervals by $\mathcal{I}_s$.

Let $\{J_j\}$ be a finite covering of C by intervals of length less than l_s. We split the set $\{J_j\}$ into finitely many groups $\mathcal{A}_s, \mathcal{A}_{s+1}, \dots, \mathcal{A}_m$, where

$$\mathcal{A}_p = \{J_j : l_{p+1} \le |J_j| < l_p\}.$$

Some of these groups may be empty. Let

$$\mathcal{M}_{s+1} = \{I \in \mathcal{I}_{s+1}; I \cap (\cup_{J\in\mathcal{A}_s} J) \ne \emptyset\}, \text{ and } M_{s+1} = \#\mathcal{M}_{s+1}.$$

We have

$$M_{s+1} \le \sum_{J\in\mathcal{A}_s} \left(\frac{|J|}{l_{s+1}} + 1\right) \le 2 \sum_{J\in\mathcal{A}_s} \frac{|J|}{l_{s+1}}.$$

Let $\mathcal{R}_{s+1} = \mathcal{I}_{s+1} \setminus \mathcal{M}_{s+1}$ and $R_{s+1} = \#\mathcal{R}_{s+1} = N_{s+1} - M_{s+1}$. We have R_{s+1} intervals from $\mathcal{I}_{s+1}$ which do not intersect intervals from $\mathcal{A}_s$.

We continue the procedure. Take all intervals from $\mathcal{I}_{s+2}$ that are contained in $\cup_{I\in\mathcal{R}_{s+1}} I$, the number of such intervals is $k_{s+1}R_{s+1}$. Let M_{s+2} of them intersect $\cup_{J\in\mathcal{A}_{s+1}} J$ and $R_{s+2} = k_{s+1}R_{s+1} - M_{s+2}$ be the number of remaining intervals.

After several steps we have R_q intervals from $\mathcal{I}_q$ that intersect no interval from $\mathcal{A}_s \cup \mathcal{A}_{s+1} ... \cup \mathcal{A}_{q-1}$. Then we have $k_q R_q$ intervals in $\mathcal{I}_{q+1}$ that intersect

no interval from $\mathcal{A}_s \cup \mathcal{A}_{s+1} \cup \ldots \cup \mathcal{A}_{q-1}$, and M_{q+1} of them intersect intervals from $\mathcal{A}_q$, where

$$M_{q+1} \le 2 \sum_{J \in \mathcal{A}_q} \frac{|J|}{l_{q+1}}. \tag{7.1}$$

Next we define $R_{q+1} = k_q R_q - M_{q+1}$. By induction

$$R_{q+1} = N_{q+1} - \sum_{r=s+1}^{q+1} \frac{M_r}{N_r} N_{q+1}.$$

If $R_{m+1} > 0$, then we can find a point in C that is not covered by the intervals from $\mathcal{A}_s \cup \ldots \cup \mathcal{A}_m$. Thus $R_{m+1} = 0$, and we get

$$\sum_{r=s+1}^{m+1} \frac{M_r}{N_r} = 1. \tag{7.2}$$

Set $b_s = \inf_{q \ge s} N_q \lambda(l_q)$. Now we use (2.5), (7.1), (7.2) to estimate the sum of $\lambda(|J_j|)$:

$$\sum_j \lambda(|J_j|) = \sum_{p=s}^{m} \sum_{J \in \mathcal{A}_p} \lambda(|J|) \ge a \sum_{p=s}^{m} \sum_{J \in \mathcal{A}_p} \lambda(l_{p+1}) \frac{|J|}{l_{p+1}} \ge$$

$$a b_s \sum_{p=s}^{m} \frac{1}{N_{p+1}} \sum_{J \in \mathcal{A}_p} \frac{|J|}{l_{p+1}} \ge \frac{a b_s}{2} \sum_{p=s}^{m} \frac{M_{p+1}}{N_{p+1}} = \frac{a}{2} b_s,$$

for any finite cover of C with $|J_j| < l_s$. This shows that

$$H_\lambda(C) \ge \frac{a}{2} \liminf_{s \to \infty} N_s \lambda(l_s).$$

Acknowledgments

This work was done when Yu.L. and E.M. visited the Universities of Aix-Marseille and Paul Sabatier. They thank the Universities for hospitality and support.

References

[1] F. Bagemihl, W. Seidel, *Some boundary properties of analytic functions*, Math. Z. **61** (1954) 186–199.

[2] A. Borichev, *On the minimum of harmonic functions*, J. Anal. Math. **89** (2003) 199–212.

[3] A. Borichev, Yu. Lyubarskii, *Uniqueness theorems for Korenblum type spaces,* J. Anal. Math. **103** (2007) 307–329.

[4] C. Cabrelli, F. Mendivil, U. Molter, R. Shonkwiler, *On the Hausdorff h-measure of Cantor sets,* Pacific J. Math. **217** (2004) no. 1, 45–59.

[5] V. Eiderman, *On a comparison between the Hausdorff measure and capacity,* (Russian) Algebra i Analiz **3** (1991) no. 6, 173–188; translation in St. Petersburg Math. J. **3** (1992) no. 6, 1367–1381.

[6] J. Garnett, D. Marshall, *Harmonic measure*, New Mathematical Monographs, 2, Cambridge University Press, Cambridge, 2005, xvi+571 pp.

[7] J. Heinonen, *Analysis on metric spaces*, Universitext, Springer, 2000.

[8] C. Horowitz, *Zeros of functions in the Bergman spaces,* Duke Math. J. **41** (1974) 693–710.

[9] J-P. Kahane, Y. Katznelson, *Sur le comportement radial des fonctions analytiques,* C. R. Acad. Sci. Paris Sr. A-B **272** (1971) A718–A719.

[10] B. Korenblum, *An extension of the Nevanlinna theory,* Acta Math. **135** (1975) no. 3-4, 187–219.

[11] N. K. Nikolskiĭ, *Selected problems of weighted approximation and spectral analysis*, Trudy MIAN **120** (1974); English translation in Proc. of the Steklov Institute of Math. **120** (1974), Amer. Math. Soc., Providence, RI (1976), 276 pp.

[12] I. I. Privalov, *Boundary properties of analytic functions*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950, 336 pp.

[13] K. Seip, *On Korenblum's density condition for the zero sequences of* $A^{-\infty}$, J. Anal. Math. **67** (1995) 307–322.

Université Aix-Marseille, 39, rue Joliot Curie, 13453, Marseille Cedex 13, France

E-mail address: borichev@cmi.univ-mrs.fr

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

E-mail address: yura@math.ntnu.no

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

E-mail address: eugnia@math.ntnu.no

Université Paul Sabatier, 31062 Touluose Cedex 9, France

E-mail address: pthomas@math.univ-toulouse.fr